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Splitting the Cartesian Point

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It is argued that the point structure of space and time must be constructed from the primitive "extensional" character of space and time. A procedure for doing this is laid down and applied to one-dimensional and two-dimensional systems of abstract "extensions." Topological and metrical properties of the constructed point systems, which differ nontrivially from the usual $\mathbb{R}$ and $\mathbb{R}^2$ models, are examined. Briefly, constructed points are associated with "directions" and the Cartesian point is split. In one-dimension each point splits into a point pair compatible with the linear ordering. An application to one-dimensional particle motion is given, with the result that natural topological assumptions force the number of "left point, right point" transitions to remain locally finite in a continuous motion. In general, Cartesian points are seen to correspond to certain filters on a suitable Boolean algebra. Constructed points correspond to ultrafilters. Thus, point construction gives a natural refinement of the Cartesian systems.

1. INTRODUCTION

The purpose of this paper is twofold: (1) To reexamine the classical or Cartesian concept of a point and to define and apply a point-construction procedure in suitably defined models of one- and two-dimensional spaces of elemental extensions, and (2) to investigate some topological and metrical properties of the constructed point systems and examine certain of their consequences.

The new point structures can be looked on as refinements of the usual continuum-based structures as a result of the "splitting" of the Cartesian points of space (and time). The nature of the splitting turns out to depend on the local "extensional" structure of the space. While the construction procedure used is based on a combination of the methods of Russell and Whitehead, the close connection with the concepts of modern set theory is brought out in Section 8. It turns out that, in a suitably defined and natural

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Boolean algebra, classical or Cartesian points are filters and constructed points free ultrafilters. This justifies the description of the constructed (or "ultrapoint") systems as refinements of the classical systems.

Interestingly, a new geometrically based concept of "state" for point particles is suggested. In one dimension, for example, each Cartesian point is replaced by a pair of ultrapoints. This can be utilized by classifying point particles in one of two states, depending on the member of the constructed pair with which it is associated. Under natural topologizations of one-dimensional space and time (itself presented as a constructed point system) a motion involving what would be discontinuous changes of state in the classical picture appears as continuous in the new structure. The requirement of continuity in these topologies imposes, however, the restriction to a locally finite number of such changes of state.

The order structure of the systems of ultrapoints suggests a very natural and geometrical way of introducing infinitesimals. This is not followed up in the present paper, as natural topologies support conventional metrics as mentioned. There appears to be scope for further developments.

2. THE BACKGROUND TO POINT CONSTRUCTION

Independently and working after their collaboration on the *Principia*, Whitehead and Russell sought to answer this question. Whitehead (1920) took the more obvious approach, relating the ideas of a point in space and an instant in time to that of idealized locations characterized as the union of certain sequences of diminishing extensions, and succeeded in recovering the point structure of four-dimensional space-time. In a more restricted context, Russell (1936) took a subtler approach and derived certain sufficient conditions for classes of overlapping events to constitute instants of time.

Both logicians baulked at the idea of using abstract extensions as the elements of their theories. [Difficult problems of reference are encountered, which are treated elsewhere (Blodwell, 1978).] But their insistence on working with actual events as the primitive terms in their theoretical constructions effectively prevented their analyses from producing anything new. In fact, it is possible, using a slightly modified form of Russell's condition and applying it in a system of abstract elemental extensions, to develop interesting refinements of the Cartesian point structures of familiar spaces. In this paper the line and plane will be treated.

Russell took as primitive the idea of events overlapping in time. His treatment did not consider relativistic effects and one assumes that his events are defined locally as observed by the same observer. His definition was based on the idea that an instant in time is essentially characterized by the set of all events occurring at that instant. He expressed this formally by means of two conditions that had to be satisfied by a set of events S_t if this was to be classed as a point in time: (1) Every event in S_t must overlap every other event; (2) any event overlapping every event in S_t must itself be a member of S_t. An ordering on the instants was defined by treating the relation "wholly preceding" between events as primitive, so that, given S_t and $S_{t'}$, the instant t' would be later than t if and only if there existed an event in S_t wholly preceding some event in $S_{t'}$.

In this paper this approach will be taken, that is, a point will be defined as a certain set of elemental extensions, but in a suitably generalized form.

The logical problems of dealing directly with a system of abstract elemental extensions requires working with a generalization of a condition satisfied by the set-theoretic construction of the natural numbers, namely that the construction of the elements of the system automatically determines their interrelations. These problems are not of present concern. Suffice it to say that the procedure for constructing such a system, called a relation space, simultaneously constructs a primitive binary asymmetric relation denoted by G. In the present context this relation can be represented as follows: aGb means: b is wholly "covered" by a with no "coincidence of boundary." This is shown diagrammatically in Fig. 1 (this representation is entirely informal in the context of this paper).

In the construction theory, once G has been defined, one has great freedom in producing axioms for different relation spaces. Of crucial significance for any spaces of physical interest is the requirement that there be no minimal elements relative to G, expressing the nonexistence of minimal extensions of space or time.

Other relations, of interest in the sequel, that can be defined in relation spaces sufficiently rich in elements are as follows:

Definition 2.1. $aSb \Leftrightarrow (\exists x)(aGx \wedge bGx)$.

Informally, "a overlaps b," represented diagrammatically in Fig. 2.

Definition 2.2. $aN_Gb \Leftrightarrow (\forall x)\neg(aGb) \wedge (bGx \Rightarrow aGx)$, so aN_Ga, but N_G is not otherwise symmetrical (see Fig. 3).

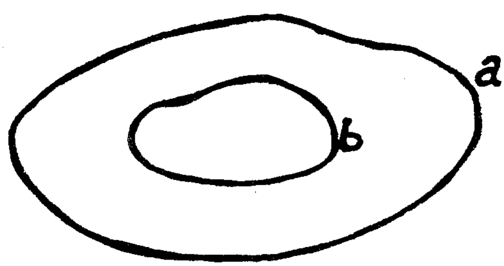

Fig. 1.

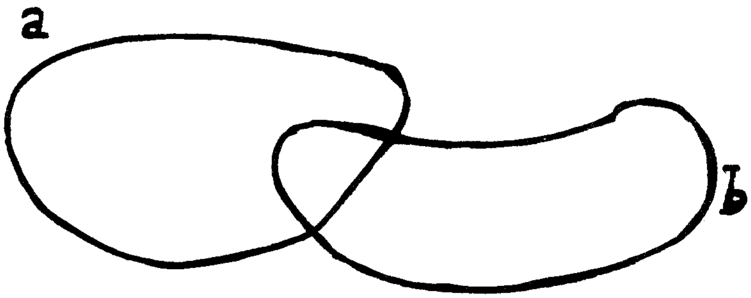

Fig. 2.

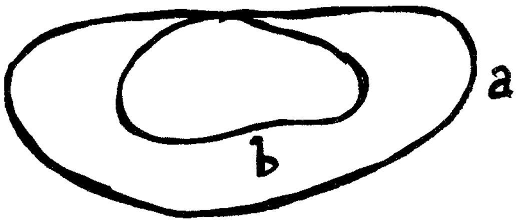

Fig. 3.

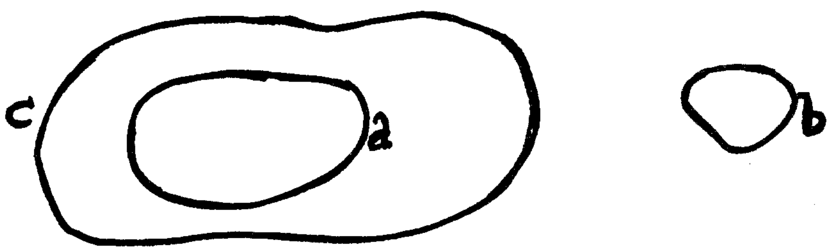

Fig. 4.

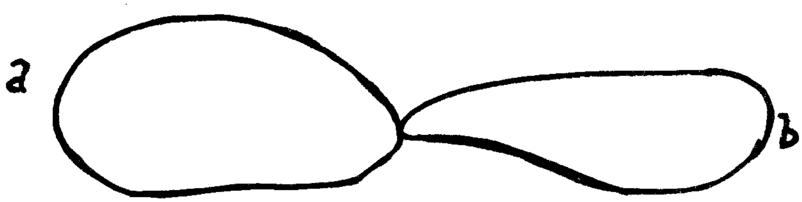

Fig. 5.

Definition 2.3. $aDb \Leftrightarrow (\exists c)(\forall x)(cGa \wedge (bGx \Rightarrow \neg(cGx))$ (see Fig. 4).

Definition 2.4. $aN_Db \Leftrightarrow (\forall c)(\exists x)(cGa \Rightarrow (cGx \wedge bGx))$ (see Fig. 5).

Note that the symmetry of N_D and D, intuitively obvious form our usual spatial concepts, would depend on whether the relevant elements were provided by the axioms of the particular relation space in question.

Some further notation is necessary.

Definition 2.5. $\vec{S}'y = \{x | xSy\}$.

Definition 2.6. $\overleftarrow{S}'y = \{x | ySx\}$.

Definition 2.7. $S''\beta = \{x | (\exists y)(y \in \beta \wedge xSy\}$.

Definition 2.8. $p'\vec{S}''\beta = \{x | (\forall y)(y \in \beta \Rightarrow xSy\}$.

It is convenient to introduce the notion of a d-set:

Definition 2.9. $A = \{x | (\forall u)(uGa \Rightarrow uGx)\}$.

The convention of associating with each elemental extension a its d-set, denoted by corresponding capital A, is retained for the sequel.

Point construction is very sensitive to the "connectedness" of elemental extensions. In this paper elemental extensions are taken to be "connected," where the precise meaning of this term will be clear when the particular models are described. The justification of this rests on the basic idea of taking elemental extensions as primitive. Then "disconnected" elemental extensions are simply distinct elemental extensions, and any concept analogous to set-theoretic union is incompatible with this primitivity.

Further, the existence of points depends on the "richness" of the elemental extensions making up a relation space. In particular, the intersection or overlapping properties are of great importance. Elemental extensions overlap in elemental extensions. Typically, in the notation of d-sets we

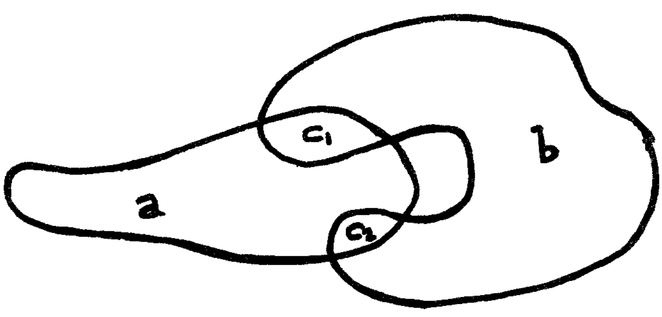

Fig. 6.

have, for given elements a and b,

$$A \cap B = \bigcup_{i \in I} C_i \wedge \bigcap_{i \neq j} C_i \cap C_j = \varnothing$$

See Fig. 6, where c_1, c_2 are elements in which a and b overlap.

Returning to Russell's definition of a point given in the introduction, it is noted that it can be formally stated

$$\alpha \text{ is a point if and only if } \alpha = p'\vec{S}''\alpha$$

The suitability for application of this definition depends on the nature of the sets in which elements are permitted to overlap.

If the axioms for a relation space permit infinite overlappings (i.e., the index set I above is infinite), then the Russell definition is unsuitable for spaces of more than one dimension. In this case it is necessary to strengthen the definition of a point. This can be done by introducing a new relation S_θ:

Definition 2.10. $xS_\theta y \Leftrightarrow (\exists z)(z \in X \cap Y \cap p'S''\theta)$.

This means that a set of elemental extensions is a point if and only if (1) every element of θ overlaps every other element of θ in elements that themselves overlap every member of θ, and (2) every element that overlaps every element of θ in elements that also overlap every element of θ itself belongs to θ.

In this paper, though the stronger definition is mentioned, we make no further use of it beyond a reference in Section 6. In this section, dealing with point construction in a two-dimensional space, it is indicated how the strengthened definition would yield points when the original Russell definition would fail to do so. It happens that, even so, the order structure of the ensuing point complexes loses any intuitive coherence, though this does not of course imply that it is ultimately without interest.

One can see even at this stage that it is not possible that two elements have a point in common unless they overlap in other elemental extensions. This means that the Cartesian idea of "elements" disjoint everywhere except at common boundary points is inadmissible here.

3. POINT CONSTRUCTION ON THE LINE

The emphasis in this paper lies on point construction and its results rather than on axiomatic development of relation spaces from their primitive origins. Hence the line will be defined as a set of elemental extensions explicitly in terms of the reals. Properly speaking, we are treating a model of the abstract relation space $\mathscr{L}_1$ defined elsewhere (Blodwell, 1978). The elements of this model of $\mathscr{L}_1$ are defined to be the finite open intervals of $\mathbb{R}$, and the naming convention as typified by

$$x \in \mathscr{L}_1 \text{ corresponds to } (x_1, x_2), \text{ where } x_1, x_2 \in \mathbb{R} \text{ and } x_1 < x_2$$

The existence of the elements of $\mathscr{L}_1$ and their order structure is thus expressed here as properties of $\mathbb{R}$.

For the relations defined in Section 2 we have the following results:

1. aGb whenever $a_1 < b_1 < b_2 < a_2$.
2. aSb whenever $(a_1, a_2) \cap (b_1, b_2) \neq \varnothing$.
3. aN_Gb whenever $a_1 = b_1 < b_2 \leq a_2$ or $a_1 \leq b_1 < b_2 = a_2$.
4. aDb whenever $(a_1, a_2) \cap (b_1, b_2) = \varnothing$ and both $a_2 \neq b_1$, $a_1 \neq b_2$.
5. aN_Db whenever $(a_1, a_2) \cap (b_1, b_2) = \varnothing$ and either $a_1 = b_2$ or $a_2 = b_1$.

The following partial ordering relations are needed:

6. xLy whenever $(x_1, x_2) \cap (y_1, y_2) = \varnothing$ and $x_2 < y_1$.
7. xRy whenever $(x_1, x_2) \cap (y_1, y_2) = \varnothing$ and $y_2 < x_1$.

The following composite relations are needed in order to characterize first and last points of an element of $\mathscr{L}_1$:

Definition 3.1. $xL|Sy \Leftrightarrow (\exists u)(xLu \cap uSy)$.

This holds in the $\mathbb{R}$ model when $(x_1, x_2) \cap (y_1, y_2) \neq \varnothing$ and $x_2 < y_2$. This gives rise to the following set:

Definition 3.2. $\{x | xL|Sy\} = L''\vec{S}'y$.

The relations $xR|Sy$ and the set $R''\vec{S}y$ are similarly defined.

$\mathbb{P}_1$: *The Point Structure of* $\mathscr{L}_1$. The "first point" of $a \in \mathscr{L}_1$ is the set α, where

$$\alpha \equiv \vec{S}'a - L''\vec{S}'a = p'\vec{S}''_{\alpha}\alpha$$

It is not difficult to verify that α is made up precisely of those elements x corresponding to (x_1, x_2) where $x \leq a_1 < x_2$. That is, α is the set of intervals overlapping a but not beginning to the left of any interval overlapping a. In the relational notation α is made up of all those x in the set

$$\vec{G}'a \cap (\vec{N}'_Ga - L''\vec{S}'a) \cap \vec{N}'_Ga \cap (S''\vec{R}'a \cap \vec{S}'a)$$

In a similar way a has a "last point" β defined by

$$\beta \equiv \vec{S}'a - R''\vec{S}'a = p'\vec{S}''_{\beta}\beta$$

Again, it may be verified that β is made up of all those intervals (x_1, x_2) where $x_1 < a_2 \leq x_2$.

Thus, given elements a and b of $\mathscr{L}_1$ where aN_Db and aLb (that is, $a_1 < a_2 = b_1 < b_2$), it follows that the common "boundary" point of their Cartesian representation is "split" into two points,

$${}_1\theta = \{(x_1, x_2)|x_1 < a_2 \leq x_2\}, \qquad \theta_1 = \{(x_1, x_2)|x_1 \leq b_1 < x_2\}$$

The points are evidently distinct, since $a \in {}_1\theta - \theta_1$ and $b \in \theta_1 - {}_1\theta$. An ordering on $\mathbb{P}_1$ may be defined in terms of the partial ordering L on $\mathscr{L}_1$.

Definition 3.3. $\alpha < \beta \Leftrightarrow (\exists x)(\exists y)(x \in \alpha \wedge y \in \beta \wedge xLy)$.

By using the completeness of $\mathbb{R}$, it may be shown that every point in $\mathbb{P}_1$ is either a "left point" (or "last point") or a "right point" or ("first point") like ${}_1\theta$ (or β) and θ_1 (or α), respectively. It follows from this that the $<$ order on $\mathbb{P}_1$ is a total order.

Evidently there is no point ξ such that ${}_1\theta < \xi < \theta_1$, for this would entail the existence of an element x where $x \in \overleftarrow{L}'a \cap \vec{L}'a$. There is no such interval. Hence ${}_1\theta$ and θ_1 are strictly consecutive.

Since each Cartesian point in $\mathbb{R}$ can be looked on as a boundary point between disjoint open intervals, it follows that to each point of $\mathbb{R}$ there corresponds such a point pair $\{{}_1\theta, \theta_1\}$. Hence $\mathbb{P}_1$ is a continuum of point pairs that are strictly consecutive in the total $<$ ordering. To illustrate, consider the Cartesian points 1 and 2. The constructed point system $\mathbb{P}_1$ splits there, giving $\{{}_11, 1_1\}$ and $\{{}_12, 2_1\}$.

Of course, ${}_11 < 1_1 < {}_12 < 2_1$, since, for example,

$$(\tfrac{1}{2}, 1) \in {}_11, \qquad (1, 1\tfrac{1}{2}) \in 1_1, \qquad (1\tfrac{1}{2}) \in 1_1, \qquad (1\tfrac{1}{2}, 2) \in {}_12, \qquad (2, 3) \in 2_1$$

4. TOPOLOGIZATION AND METRIZATION OF $\mathbb{P}_1$

Two topologies are considered.

4.1. The Open-Interval Topology

Let the set of left points of $\mathbb{P}_1$ be denoted by $\mathbb{P}_1^l$ and the set of right points by $\mathbb{P}_1^r$.

Define open intervals of left (right) points as follows:

Definition 4.1. $({}_1\alpha, {}_1\beta) = \{{}_1\theta | \alpha < \theta < \beta\}$.

Definition 4.2. $(\alpha_1, \beta_1) = \{\theta_1 | \alpha < \theta < \beta\}$.

Let $I_1(I_2)$ be the set of all open intervals of $\mathbb{P}_1^l$ ($\mathbb{P}_1^r$). Then $I_1 \cup I_2$ is the basis for a topology T on $\mathbb{P}_1$ that induces the Euclidean topology on both $\mathbb{P}_1^l$ and $\mathbb{P}_1^r$. First it is noted that T is normal. To see this, it is observed that any set closed in T is the union of sets closed in $\mathbb{P}_1^l$ and $\mathbb{P}_1^r$, and conversely that the union of such sets is closed in T. Since the Euclidean topology is normal, disjoint sets closed in T can be contained in disjoint sets open in T.

Since T is normal, there exists a metric function $d: \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}$ that induces T on $\mathbb{P}_1$.

The condition

$$d({}_1\mu, {}_1\nu) = d(\mu_1, \nu_1) = |\mu - \nu| \qquad \text{for any} \quad \mu, \nu \in \mathbb{R}$$

is imposed so that d may induce the Euclidean metric on $\mathbb{P}_1^l$ and $\mathbb{P}_1^r$.

Clearly the conditions for a metric function require that $d({}_1\mu, \mu_1) > 0$, since ${}_1\mu$ and μ_1 are distinct points. The question arises; "Can this length vary as μ varies over $\mathbb{R}$?" To give a partial answer to this it is necessary to consider ways of defining $d({}_1\mu, \nu_1)$ and $d(\mu_1, {}_1\nu)$. Note first that $\lim_{\mu \to \nu} d(\mu_1, {}_1\nu) \neq 0$ if this limit exists, since ${}_1\nu$ is not a limit point of the set (μ_1, ν_1). Thus, it is not permitted to define $d(\mu_1, {}_1\nu) = |\nu - \mu|$ in general. Similarly, $d({}_1\mu, \nu_1) = |\nu - \mu|$ is also ruled out. In giving the definition following, the idea is that to "move" a point from ${}_1\nu$ to μ_1 a "flip over" from a left to a right point must be made somewhere between μ and ν, and similarly for the distance between ν_1 and ${}_1\mu$. First, for brevity the function f is defined, where $f: \mathbb{R} \to \mathbb{R}$:

Definition 4.3. $d({}_1\mu, \mu_1) = f(\mu)$ for all $\mu \in \mathbb{R}$.

Suppose that the definition of d is completed by:

(a) f is continuous.
(b) $d({}_1\mu, \nu_1) = d(\mu_1, {}_1\nu) = |\nu - \mu| + \inf_{\mu \leq \theta \leq \nu} f(\theta)$

Under these assumptions we have the following result:

Theorem. f is constant on $\mathbb{R}$.

Proof. If f is not constant, then for some α and $\partial \in \mathbb{R}$, if $\theta_1 \in [\alpha - \partial, \alpha]$ and $\theta_2 \in [\alpha, \alpha + \partial]$, then $f(\theta_1) < f(\theta_2)$ or $f(\theta_1) > f(\theta_2)$. This follows from the continuity of f and considerations similar to those used in the proof of the Intermediate Value Theorem. To illustrate the argument, the former case is considered. Take μ, ν, ρ, where $\mu < \nu < \alpha < \rho$, sufficiently close to α such that

$$\tfrac{1}{2}[\inf_{\mu \leq \theta \leq \nu} f(\theta) - f(\rho)] > \rho - \nu$$

This contradicts the triangle inequality for d, however, since

$$d({}_1\mu, \nu_1) > d({}_1\mu, {}_1\rho) + d({}_1\rho, \rho_1) + d(\rho_1, \nu_1)$$

We have

$$\nu-\mu+\inf_{\mu\le\theta\le\nu} f(\theta)-[\rho-\mu+f(\rho)+\rho-\nu)$$

$$=\inf_{\mu\le\theta\le\nu} f(\theta)+2\nu-2\rho-f(\rho)$$

$$=\inf_{\mu\le\theta\le\nu} f(\theta)-f(\rho)+2(\nu-\rho)>0$$

by the above inequality. Hence, d as defined above is not a metric. To avoid the contradiction, it is necessary to have $\inf_{\mu\le\theta\le\nu} f(\theta)=f(\rho)$ whatever the values of μ, ν, and ρ, subject to the above assumption, so that f is constant in the neighborhood of α. By a suitable generalization of this argument it can be concluded that a continuous f must be constant everywhere on $\mathbb{R}$.

The most natural metric inducing T is then given as follows:

Definition 4.4. $d({}_1\mu, {}_1\nu)=d(\mu_1, \nu_1)=|\nu-\mu|$

$$d({}_1\mu, \nu_1)=d(\mu_1, {}_1\nu)=|\nu-\mu|+k$$

for all μ, $\nu\in\mathbb{R}$, where k is the constant "flipover" distance $d({}_1\mu, \mu_1)$ for all μ.

4.2. The Inner-Interval Topology

Associate with each element x of L_1 the set of points X, where for any point $p\in\mathbb{P}_1$, $p\in X\Leftrightarrow x\in p$ (the term X must be distinguished by context from the d-set, also denoted by X and associated with x, defined in Definition 2.9).

A topology F is given by taking the collection of such sets together with the null set $\varnothing$ and the universal set $\mathbb{P}_1$ as a basis.

A typical basis set of F, with (α, β) corresponding to x (see Section 3), is the "inner interval" $[\alpha_1, {}_1\beta]$. The square brackets indicate that α_1 and ${}_1\beta$, the first and last points of x, belong to the basis set. Note that such sets are in a sense the analogues of the closed intervals in $\mathbb{R}$, since ${}_1\alpha$ and β_1 do not belong to the closure of $[\alpha_1, {}_1\beta]$. Indeed, $[\alpha_1, {}_1\beta]$ is itself closed, so that F is then zero-dimensional. Since F is clearly a Lindelof space, it is strongly zero-dimensional, but is not extremely disconnected; although it is not difficult to see that F is normal, it is not second countable, and therefore not metrizable under the usual assumptions of set theory.

5. AN APPLICATION TO ONE-DIMENSIONAL MOTION

Denote a one-dimensional space of point pairs by S_1, and let it be endowed with the open-interval topology H described in Section 4.1. Denote the one-dimensional time of "instant pairs" by T and let it be endowed with the inner-interval topology F described in Section 4.2. A particle P will be considered as a point particle in S_1 by being associated with the constructed points of S_1, and similarly an instant of time will be either a "left instant" or a "right instant." A pseudometric will be imposed on T as follows:

Definition 5.1. $d({}_1\alpha, \alpha_1) = 0$ for all α

$$d({}_1\alpha, {}_1\beta) = d(\alpha_1, \beta_1) = d({}_1\alpha, \beta_1) = d(\alpha_1, {}_1\beta) = |\alpha = \beta|$$

Each point of S_1 is of course either a left or a right point, and so two kinds of particle are considered, namely left-pointing or right-pointing. A possible motion of a particle P will be a continuous map $f\colon T \to S_1$ such that $f({}_1t) = f(t_1) = {}_1\alpha$ for a left-pointing particle and $g({}_1t) = g(t_1) = \alpha_1$ for a right-pointing particle for classical times t and points α. One can think of ${}_1t$ as the instant of arrival of P at ${}_1\alpha$ and t_1 as the instant of departure, with zero time between these instants. For such motions the use of the instant-pair time system introduces nothing new, but a more interesting possibility arises when transformations between left- and right-pointing particles are considered.

A transformation of a left-pointing to a right-pointing particle occurring during a motion defined as follows:

$$h\colon T \to S_1 \qquad \text{where} \quad \begin{array}{ll} h({}_1t) = h(t_1) = {}_1\alpha, & t < s \\ \left.\begin{array}{l} h({}_1s) = {}_1\beta \\ h(s_1) = \beta_1 \end{array}\right\}, & t = s \\ h({}_1t) = h(t_1) = \alpha_1 & t > s \end{array}$$

This motion is sequentially continuous at both the left and right points ${}_1\beta$ and β_1 associated with such a transformation over the point-instant interval ${}_1s$ and s_1 associated with the classical instant s. To see this, consider any pair of sequences $\{{}_1t^{(i)}\}$, $\{t_1^{(i)}\}$, where $t^{(1)} < t^{(2)} \cdots < t^{(i)} < t^{(i+1)} \cdots < s$, which are such that $\lim_{i\to\infty} {}_1t^{(i)} = {}_1s$ and $\lim_{i\to\infty} t_1^{(i)} = {}_1s$ in the F topology on T. However, $\lim_{i\to\infty} h({}_1t^{(i)}) = {}_1\beta$ and also $\lim_{i\to\infty} h(t_1^{(i)}) = {}_1\beta = h({}_1s)$. Similarly for sequences $\{{}_1u^{(i)}\}$, $\{u_1^{(i)}\}$, where $u^{(1)} > u^{(2)} > \cdots u^{(i)} > u^{(i+1)} \cdots > s$, and $\lim_{i\to\infty} u^{(i)} = s$ it follows that $\lim_{i\to\infty} ({}_1u^{(i)}) = s_1$ and $\lim_{i\to\infty} h({}_1u^{(i)}) = \beta_1 = h(s_1)$; and also $\lim_{i\to\infty} (u_1^{(i)}) = s_1$ and $\lim_{i\to\infty} h(u_1^{(i)}) = \beta_1 = h(s_1)$. Now, h is clearly sequentially continuous at other points, so that the motion of the left-pointing particle up to ${}_1\beta$, and its transformation to a right-pointing particle expressed by its occurrence at β_1, together with its subsequent motion to the right of β_1, is everywhere sequentially continuous. The number of such transformations must, however, be locally finite. To see this, consider the sequence $\{t^{(n)}\}$, where $t^{(n)} < t^{(n+1)}$ and $\lim_{i\to\infty} t^{(n)} = s$. Now consider a function g such that

$$g({}_1t^{(2n-1)}) = {}_1\alpha^{(2n-1)}; \qquad g(t_1^{(2n-1)}) = \alpha_1^{(2n-1)}$$

and

$$g({}_1t^{(2n)}) = \alpha_1^{(2n)}; \qquad g(t_1^{(2n)}) = {}_1\alpha^{(2n)} \qquad \text{for each } n$$

and that

$$g({}_1t) = {}_1\alpha \qquad \text{for} \quad t < t^{(1)}$$
$$g({}_1t) = g(t_1) = \alpha_1 \qquad \text{for} \quad t^{(2n-1)} < t < t^{(2n)}$$
$$g({}_1t) = g(t_1) = {}_1\alpha \qquad \text{for} \quad t^{(2n)} < t < t^{(2n+1)}$$

i.e., a left-pointing particle arrives at ${}_1\alpha^{(1)}$ from the left and is transformed into a right-pointing particle and moves on to the right, transforming back to a left-pointing particle at the ${}_1t^{(2)}$, $t_1^{(2)}$ point pair, moving on, and transforming infinitely often as $t \to s$. Supposing that the particle exists at the classical time s, it does so as either a left- or right-pointing particle. For definiteness take $g({}_1s) = {}_1\gamma$, where $\lim_{n\to\infty} \alpha^{(n)} = \gamma$. Now define the subsequences $\{t^{(2n)}\}$, $\{t^{(2n+1)}\}$. Then

$$\lim_{n\to\infty} {}_1t^{(2n)} = \lim_{n\to\infty} t_1^{(2n+1)} = {}_1s$$

But

$$\lim_{n\to\infty} ({}_1t^{(2n+1)}) = {}_1\gamma = g({}_1s), \qquad \lim_{i\to\infty} g(t_1^{(2n+1)}) = \gamma_1 \neq g(s_1)$$

so that g cannot be sequentially continuous at s. This precludes the possibility that a sequentially continuous motion can contain any pattern of infinitely repeated transformations in any finite region of the one-dimensional space.

6. POINT CONSTRUCTION IN THE PLANE $\mathscr{L}_2$

As in the case of the line $\mathscr{L}_1$ it would be inappropriate to write down a set of abstract axioms. The system is presented in terms of a concrete model. The elemental extensions of $\mathscr{L}_2$ are defined as the open interiors of a certain set B of closed and bounded one-parameter paths in $\mathbb{R}^2$. Each such path is looked on as an image of the unit interval $I = [0, 1]$, with first and last points identified satisfying certain finiteness conditions. More precisely, if $f \in B$, then the following holds:

Definition 6.1. (a) $f(0) = f(1)$.

(b) f is left and right differentiable everywhere, with inequality at most at a finite number of points.

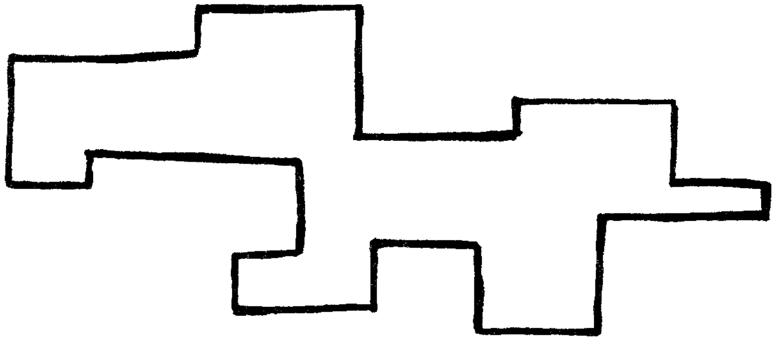

Fig. 7.

(c) If P is the set of lines in $\mathbb{R}^2$, then for each $l \in P$, $f(I) \cap l$ is either: (i) null, (ii) a finite number of discrete points, or (iii) a finite number of discrete closed intervals on a finite number of lines.

The strong finiteness conditions are chosen to ensure that the point complexes replacing each Cartesian point of the plane are locally ordered, in a sense of the term order to be made clear.

Of course it is possible to find many sets of closed paths in $\mathbb{R}^2$ satisfying these requirements. The simplest choice for B would be to restrict the range of f to line segments of the form

$$\{(k_1, y) | y_1 \leq y \leq y_2\} \qquad \text{and} \qquad \{(x, k_2) | x_1 \leq x \leq x_2\}$$

with k_1 and k_2 constants. That is, the elemental extensions would be the open interiors of regions typified as shown in Fig. 7. It is of interest to examine the point structure for such a system. To do so, consider the points associated with the origin $(0, 0)$ of the system $\mathbb{R}^2$ used for defining the model. Let $S_\varepsilon^{(1)}$ be the element corresponding to the open interior of a square with corners at $(0, 0)$, $(\varepsilon, 0)$, $(\varepsilon, \varepsilon)$, and $(0, \varepsilon)$. Let S_2, S_3, S_4 be similar squares drawn in the second, third, and fourth quadrants. Then it is easy to see that $\theta_1 = \{x | (\exists \varepsilon)(S_\varepsilon^{(1)} \subset x)\}$ is such that, for all $x, y \in \Theta_1$, $xS_{\theta_1}y$ and that θ_1 is a point. Similarly, θ_2, θ_3, θ_4 are points with $S_\varepsilon^{(2)}$, $S_\varepsilon^{(3)}$, $S_\varepsilon^{(4)}$ replacing $S_\varepsilon^{(1)}$ in the definition of θ_1.

In an order-theoretic sense this four-point cluster associated with the single Cartesian point at the origin is similar to the cluster obtained by taking the Cartesian product $\mathbb{P}_1 \times \mathbb{P}_1$. There is a subtle difference that may be informally expressed diagrammatically as shown in Fig. 8. Figure 8a expresses the structure for the simple example of a two-dimensional space

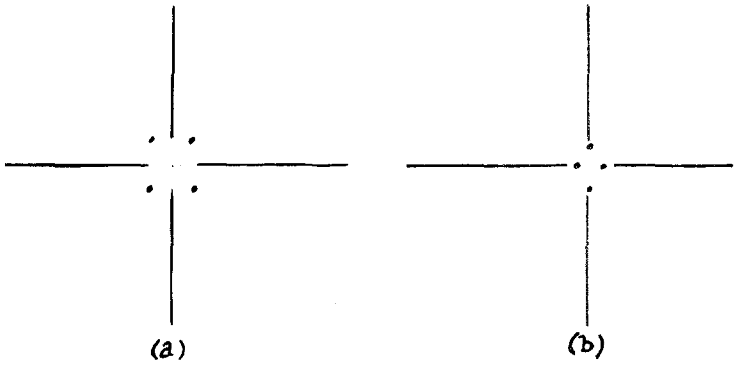

Fig. 8.

of elemental extensions considered above; Fig. 8b illustrates the structure of each point cluster of $\mathbb{P}_1 \times \mathbb{P}_1$. The former case illustrates the association of each point with a set of "two-dimensional" extensions; the latter illustrates the case where the elemental extensions belong to the constituent systems only.

This kind of distinction vanishes when B is enlarged, say to admit all possible line segments of $\mathbb{R}^2$. In this case the point cluster associated with each Cartesian point is a system of point pairs order-isomorphic to the circle (Fig. 9). To make the structure of each constructed point explicit, choose as an example that pair $\theta_m^{(\pm)}$ associated with the line $y = mx$ in the first quadrant ($m > 0$; $x > 0$). Let $m_{\varepsilon,\eta}^{\pm}$ denote, respectively, the elemental extensions corresponding to the open regions bounded by

$$\left\{(x, mx) \middle| 0 \leq x \leq \frac{\varepsilon}{m}\right\}, \quad \left\{(x, (m+\eta)x) \middle| 0 \leq x \leq \frac{\varepsilon}{m+\eta}\right\},$$

$$\left\{(x, \varepsilon)\, \frac{1}{m+\eta} \leq x \leq \frac{\varepsilon}{m}\right\}$$

and

$$\left\{(x, mx) \middle| 0 \leq x \leq \frac{\varepsilon}{m}\right\}, \quad \left\{(x, (m-\eta)x) \middle| 0 \leq x \leq \frac{\varepsilon}{m-\eta}\right\},$$

$$\left\{\left(\frac{\varepsilon}{m}, y\right) \frac{(m-\eta)\varepsilon}{m} \leq y \leq 1\right\}$$

where $|\eta|$ is small (see Fig. 10). The points θ_m^+, θ_m^- are defined as follows.

Definition 6.2. $\theta_m^{\pm} = \{x \mid \exists \varepsilon, \eta > 0 x \supset m_{\varepsilon,\eta}^{\pm}\}$.

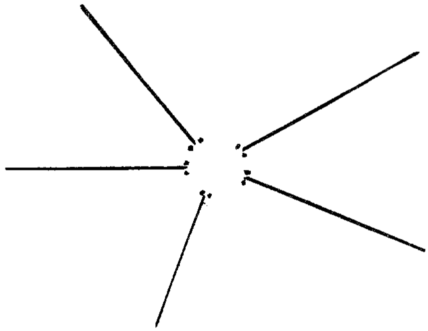

Fig. 9.

As indicated, this pair is associated with the *single* line segment, namely that part of $y = mx$ defined for $0 \leq x \leq \varepsilon/m$. Were the version of the plane to be restricted by discrete choices for η, a similar situation to the first version considered would recur, with each point being associated with a "consecutive" ε, η pair. The richest version of the plane will be obtained by taking a maximal B compatible with Definition 6.1. To express the order structure of the cluster replacing each Cartesian point, it is necessary to

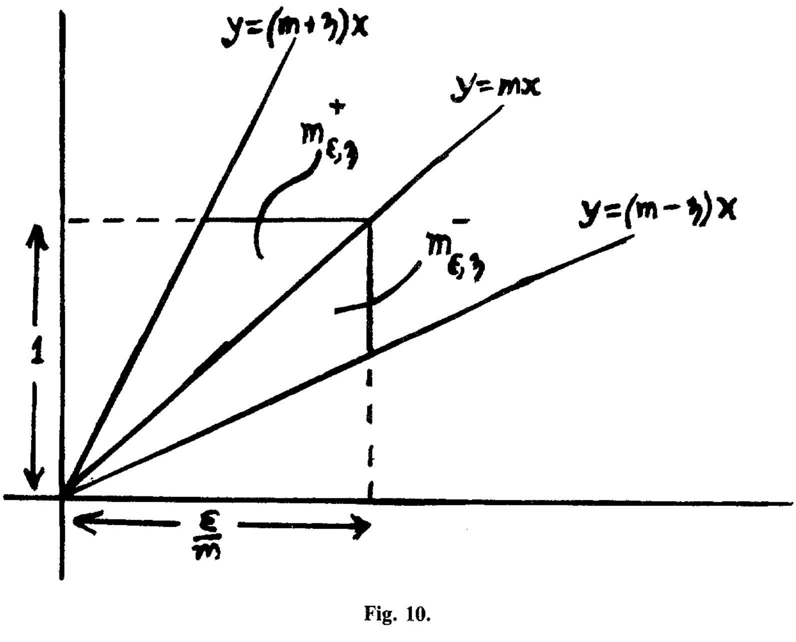

Fig. 10.

recall the definition of an η_1-set. A linearly ordered set A is an η_1-set if for all sequences $\{x_1, x_2, \dots\}$ and all elements x such that $x_1 < x_2 < \cdots < x$, there must exist a y where $x_i < y < x$ for all x_i. That is, no denumerable sequence can "converge" to any element in the natural topology. An η_1-circle will be defined by taking a linearly ordered η_1-set with first and last elements, identifying these end elements and adapting the definition of the ordering. The point cluster replacing the Cartesian point when B is taken maximally will be such an η_1-circle of point pairs.

To end this section, the effect of relaxing the finiteness conditions of Definition 6.1 is noted. If infinite overlappings are permitted, then $\mathscr{L}_2$ contains elements corresponding to the set defined according to the following inequalities:

$$1 > x > 0; \qquad 2 > y \quad \text{and} \quad y > \sin\frac{1}{x} \quad \text{if} \quad \sin\frac{1}{x} > 0, \qquad \text{otherwise} \quad y > 0$$

Then $(\forall x)(x \in \theta_m^+ \Rightarrow ySx)$, in particular. But clearly $y \notin \theta_m^+$ by the definition of this set (Definition 6.2). Hence θ_m^+ cannot be a point according to the Russell definition. However, θ_m^+ is preserved as a point according to the strengthened definition (Definition 2.10), since $\neg(yS_\theta x)$ for any $x \in \theta_m^+$.

As might be expected, however, the introduction of such elemental extensions into the relation space leads to an enriching of the point structure. For example, the line $y = \sin(1/x)$ may replace the line $y = mx$ in an adaptation of the argument leading to the definition of the points θ_m^+ and θ_m^-. The new points cannot, however, be incorporated either into the order structure of the circle of point pairs or to the η_1-circle of point pairs referred to above.

7. TOPOLOGIZATION AND METRIZATION OF THE PLANE

The point system $\mathbb{P}_2$ for the plane will be taken to be that arising from the second of the three versions of the plane considered in Section 6. That is, each Cartesian point of $\mathbb{R}^2$ is replaced by a cluster of point pairs having the order structure of the circle, with the consecutive ordering of the members of a pair compatible with the local ordering on the circle. The elemental extensions in the model are bounded by line segments in $\mathbb{R}^2$. Fixing some particular line l in the set P of all lines in $\mathbb{R}^2$ and taking a point p lying on this line, we assign the values 0_1, 0_2, 0_3, 0_4 to the four points, in the point cluster associated with p, associated with l (see Fig. 11). The set of lines P_p passing through p can be indexed relative to this reference line by the parameter θ, where $0 < \theta < \pi$. At p each of these lines is associated with a four-point subcluster, itself associated with θ and identified as above.

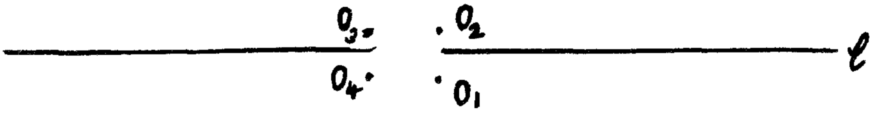

Fig. 11.

Typically these four points will be given the coordinates θ_1, θ_2, θ_3, θ_4 as in Fig. 12.

The complete coordination of $\mathbb{P}_2$ assigns to each point α a triple of numbers (x_1, x_2, x_3), where x_1 and x_2 are the Cartesian coordinates (x_1, x_2) of the point p corresponding to the cluster of which α is a member; and x_3 is of the form θ_1, θ_2, θ_3, θ_4 as described.

Equivalence relations L_1, L_2, on $\mathbb{P}_2$ are defined as follows:

Definition 7.1.

(a) $\alpha L_1 \alpha^1$ if and only if both $x_3^1 = x_3$ and (x_1, x_2), (x_1^1, x_2^1) each belong to the same line segment of $\mathbb{R}^2$.

(b) $\beta L_2 \beta^1$ if and only if $y_1 = y_1^1$, $y_2 = y_2^1$, $y_3 = y_3^1 = \theta_i$ for the *same i*, for $0 \leq \theta < \pi$.

The topology Q on P_2 is now defined:

Definition 7.2. A is open in Q if and only if the intersection of A with both the L_1 and L_2 equivalence classes of $\mathbb{P}_2$ are open in their respective Euclidean topologies.

For each individual cluster the topology induced by Q on the proper subintervals in the circular ordering is locally homeomorphic to the open-interval topology T on $\mathbb{P}_1$.

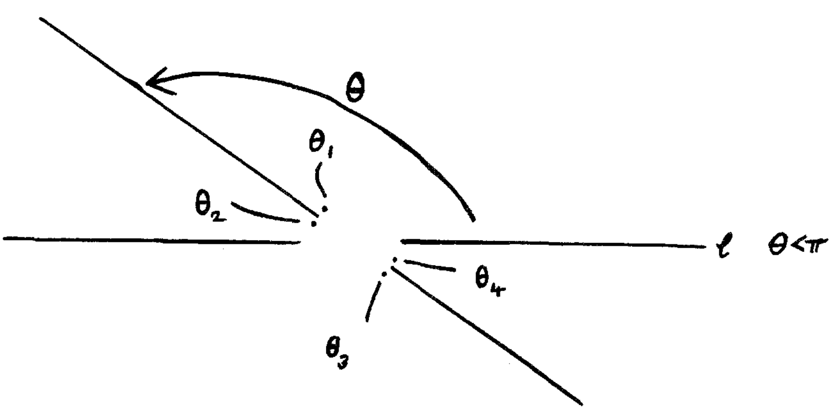

Fig. 12.

The topological space $(\mathbb{P}_2, Q)$ is metrizable. The metric d is realized as follows, where d_e denotes the Euclidean metric on $\mathbb{R}^2$.

Definition 7.3(i). $d(\alpha, \alpha^1) = d_e(p_\alpha, p_\alpha^1) \Leftrightarrow \alpha L \alpha^1$ where p_α and p_α^1 are the Cartesian points replaced by the clusters to which α and α^1, respectively, belong.

Now the "flipover" distances $d(\theta_1, \theta_2)$ and $d(\theta_3, \theta_4)$ can be shown constant on each point cluster by an argument similar to that of the Theorem in Section 4; and by an extension of that argument to the two lines of point pairs associated with each Cartesian line segment, this constant is the same for every such cluster. Let k denote this constant.

Definition 7.3(ii).

$$
\left.\begin{aligned}
d(\alpha_i, \alpha_j) &= |\alpha - \alpha'|, && i = j = 1, 2, 3, 4\\
&= |\alpha - \alpha'| + k, && i \neq j \text{ and either } i, j = 1, 2 \\
& && \text{or } i, j = 3, 4,\\
&= \pi - |\alpha - \alpha'|, && i = 1, j = 3; \text{ or } i = 2, j = 4\\
&= \pi - |\alpha - \alpha'| + k && i = 1, j = 4; \text{ or } i = 2, j = 3
\end{aligned}\right\} \quad \text{where } p_\alpha = p_{\alpha'}
$$

It remains to define the distance between nonequivalent points in different clusters. Suppose μ_i and ν_j belong to different clusters, so $p_\mu \neq p_\nu$. Let $\nu(\mu_i)$ denote that point of the cluster associated with p_ν that is L-equivalent to μ_i; then we have the following result:

Definition 7.3(iii). $d(\mu_i, \nu_j) = d_e(p_\mu, p_\nu) + d(\nu_j, \nu(\mu_i))$.

The symmetry of d requires that $d(\nu_i, \nu(\mu_i)) = d(\mu_i, \mu(\nu_j))$. This gives an alternative argument for the universality of k.

The topology Q has some interesting points of comparison with the fine topology introduced by Zeeman (1967) for space-time. It will be recalled that under the latter topology the image of any continuous map of the unit interval into Minkowski space-time is piecewise linear. The present situation is somewhat different, in that the continuous image of I would contain precisely one point of $\mathbb{P}_1$ from each of a set of Cartesian points. Under Q this latter set must be piecewise linear also. Thus, for example, if the Cartesian set is chosen to be the curve $y - x^2 = 0$ as between $(0, 0)$ and $(1, 1)$, and for each point p on this curve the constructed point θ_i, say, is selected by requiring that, e.g., $\theta = \tan^{-1} (dy/dx)_p$, then this map is nowhere continuous. Continuity requires that the image consists of an L-equivalent set of points of $\mathbb{P}_1$ selected from clusters corresponding to Cartesian points lying on line segments. The "linearity" here seems to be built in *ab initio*, but this is not really so, since no absolute characterization of "straightness" can be made prior to the introduction of the metric function. The feature is simply a result of the *order structure* of the space of lines at each Cartesian point of $\mathbb{R}^2$. When the richer version of the plane is considered, leading to the η_1-set structure of the point clusters, the topology analogous to Q is no longer second-countable. There is, however, the suggestion of a new approach to the definition of infinitesimals in a much more "concrete" way than is provided, for example, by an abstract treatment of concurrent relations (Robinson, 1966). Indeed, the third version of the plane considered in Section 5 admits a concrete realization of such relations.

8. POINTS AND ULTRAFILTERS

A Boolean algebra $(\mathbb{B}, +, \cdot, ')$ on a relation space $\mathcal{L}$ of elemental extensions is defined as follows:

Definition 8.1. First, $\mathbb{B} = \{\mathcal{A} | x \in \mathcal{A} \Leftrightarrow X \subseteq \mathcal{A}\}$, where the notation utilizes Definition 2.9.

The operations $+$ and $\cdot$ are defined with $'$, complementation, as follows:

Definition 8.2. $\mathcal{A} + \mathcal{B} = \{x | X \subset S''(\mathcal{A} \cup \mathcal{B})\}$ (see Def. 2.7)

$$\mathcal{A} \cdot \mathcal{B} = \mathcal{A} \cap \mathcal{B}$$

$$\mathcal{A}' = \{x | X \cap \mathcal{A} = \varnothing\}$$

Recall that an ultrafilter $\mathbb{U}$ is defined on a Boolean algebra $\mathbb{B}$ as follows:

1. If $\mathcal{A} \in \mathbb{U}$ and $\mathcal{B} \supset \mathcal{A}$, then $\mathcal{B} \in \mathbb{U}$.
2. If $\mathcal{A}$, $\mathcal{B} \in \mathbb{U}$, then $\mathcal{A} \cap \mathcal{B} \in \mathbb{U}$.
3. $\varnothing \in \mathbb{U}$. (Points 1–3 characterize $\mathbb{U}$ as a filter on $\mathbb{B}$.)
4. If $\mathcal{A} \in \mathbb{U}$, then $\mathcal{A} \notin \mathbb{U}$, and if $\mathcal{A} \notin \mathbb{U}$, then $\mathcal{A}' \in \mathbb{U}$.

To each Cartesian point in $\mathbb{R}$ and $\mathbb{R}^2$ there corresponds a filter in the corresponding Boolean algebra of elemental extensions. Further, to each point θ of $\mathbb{P}(\mathcal{L})$ there corresponds an ultrafilter of $\mathbb{B}$. To see this, define a family $\mathbb{U}_\theta$ corresponding to θ by the rule

$$\mathcal{A} \in \mathbb{U}_\theta \Leftrightarrow \mathcal{A} \cap \theta \neq \varnothing$$

$\mathbb{U}_\theta$ then satisfies 1–4 above, where 2 follows from the fact that if $x \in \mathcal{A} \cap \theta$ and $y \in \mathcal{B} \cap \theta$, then xSy by definition of θ. By the axioms governing both $\mathcal{L}_1$ and $\mathcal{L}_2$ there exists a v such that $V = X \cap Y$. Hence, by definition of $\mathbb{B}$, $v \in \mathcal{A} \cap \mathcal{B}$. But $v \in \theta$ also, so $(\mathcal{A} \cap \mathcal{B}) \cap \theta \neq \varnothing$. This ensures $\mathcal{A} \cap \mathcal{B} \in \mathbb{U}_\theta$. The other properties are immediate consequences of the definitions. Hence

$\mathbb{U}_\theta$ is an ultrafilter. It is a free ultrafilter, since the point θ is such that if its members are indexed in some set S, then $\bigcap_{i \in S} X_i = \varnothing$. This expresses the fact that there are assumed to be no minimal spatial extensions.

It must be noted that not every free ultrafilter defines a point. The assumption preventing this is embodied in Definition 6.1. Briefly, it is based on the requirement that elemental extensions may only intersect in a finite number of elemental extensions. An assumption of this kind is essential if any kind of coherence is to be a property of the order structure of the point clusters associated with each Cartesian point. The underlying consideration here is also evident in the one-dimensional case, where it amounts to a refusal to count, for example, a pair of disjoint intervals as corresponding to a third elemental extension in addition to those corresponding to the individual intervals. If this condition were relaxed, the simple two-point structure of the point clusters in the one-dimensional case would also be lost. It is arguable that the assumptions actually made express something of the nature of what elemental extensions "really" are, and that the consequent coherence of the clusters is natural rather than arbitrary.

Because of the connection between free ultrafilters and constructed points, it is suggested that the latter be termed "ultrapoints."

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